Programlama -1

“Differential Equations”

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Differential Equations

A differential equation (DE) may be defined as an equation involving one or more derivatives of an unknown dependent variable or several variables with respect to one or more independent variable or variables.

Linear DE versus Non-Linear DE

A *linear differential equation* is one in which the dependent variable and its derivatives with respect to the independent variable are of the first degree and all multiplicative factors are either constants or functions of the independent variable. An example follows.

$$2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = \sin 4t$$

Two Examples of Non-Linear Differential Equations

$$y^2 \frac{dy}{dt} + y = 10$$

$$\left(\frac{dy}{dt}\right)^2 + 5y = 20$$

Ordinary DE versus Partial DE

The preceding equations have been ordinary types since the dependent variable was a function of only one independent variable. An example of a *partial differential equation* follows.

$$\frac{\partial^2 y}{\partial x^2} + a\frac{\partial^2 y}{\partial t^2} = b$$

Continuous-Time versus Discrete-Time

The preceding definitions relate to *continuous-time* or *"analog"* systems. However, the same forms may be adapted to *discrete-time* or "digital" systems. In such cases, the equations are generally known as *difference equations.* Most numerical methods involve approximating differential equations as difference equations.

Boundary Conditions or Initial Conditions

The solution of an *N*th order DE usually involves *N* arbitrary constants. These constants are determined from the *boundary conditions.* When these conditions are specified as the initial value of the function and the first *N-1* derivatives, they are called *initial conditions*.

Example 9-1. Classify the following DE in several ways:

$$t^2 \frac{d^2 y}{dt^2} + 2t\frac{dy}{dt} + 5y = e^{-2t}$$

The DE is *linear* since none of the coefficients are functions of *y* and there are no higher degree terms in *y* or its derivatives.

The DE is an *ordinary* type since *y* is a function only of *t*.

Constant Coefficient Linear Ordinary Differential Equation (CCLODE)

$$b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \ldots . + b_1 \frac{dy}{dt} + b_0 y = f(t)$$

Example 9-2. Classify the DE below.

$$3\frac{d^4y}{dt^4}+5\frac{d^3y}{dt^3}+7\frac{d^2y}{dt^2}+8\frac{dy}{dt}+4y=\cos 5t+t^2$$

This DE is a CCLODE type.

It is a 4th order DE.

Simple Integrable Forms

$$b_k \frac{d^k y}{dt^k} = f(t)$$

In theory, this equation may be solved by integrating both sides *k* times. It may be convenient to introduce new variables so that only first derivative forms need be integrated at each step.

Example 9-3. An object is dropped from a height *h* at *t* = 0. Determine velocity *v* and displacement *y*.

$$\frac{dv}{dt} = g \qquad dv = g\,dt$$

$$v = gt + C_1 \qquad C_1 = 0$$

$$v = gt$$

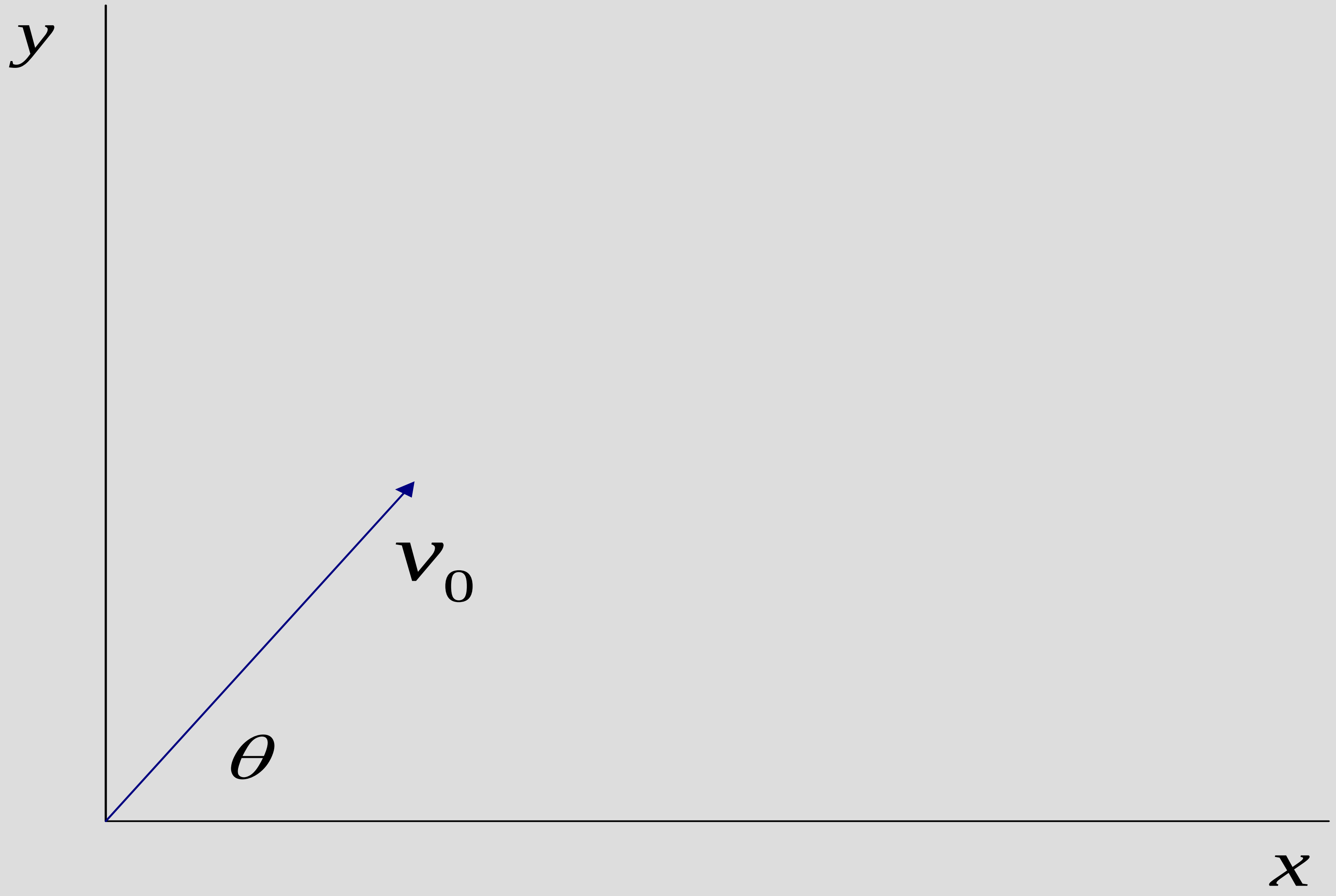
Example 9-3. Continuation.

$$\frac{dy}{dt} = gt \qquad dy = gtdt$$

$$y = \frac{1}{2}gt^2 + C_2 \qquad C_2 = 0$$

$$y = \frac{1}{2}gt^2$$

Example 9-4. Consider situation below and solve for velocity and displacement in both *x* and *y* directions.



Example 9-4. Continuation.

$$v_y(0) = v_0 \sin\theta$$

$$\frac{dv_y}{dt} = -g$$

$$v_y = -gt + C_1 \qquad C_1 = v_0 \sin\theta$$

$$v_y = -gt + v_0 \sin\theta$$

Example 9-4. Continuation.

$$y(0) = 0$$

$$\frac{dy}{dt} = v_y = -gt + v_0 \sin\theta$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + C_2$$

$$C_2 = 0$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t$$

Example 9-4. Continuation.

$$v_x(0) = v_0 \cos\theta$$

$$\frac{dv_x}{dt} = 0$$

$$v_x = C_3 \qquad C_3 = v_0 \cos\theta$$

$$v_x = v_0 \cos\theta$$

Example 9-4. Continuation.

$$x(0) = 0$$

$$\frac{dx}{dt} = v_x = v_0 \cos\theta$$

$$x = (v_0 \cos\theta)t + C_4$$

$$C_4 = 0$$

$$x = (v_0 \cos\theta)t$$

Constant Coefficient Linear Ordinary Differential Equations (CCLODE)

$$b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + ... + b_1 \frac{dy}{dt} + b_0 y = f(t)$$

$$y = y_h + y_p$$

The *general solution* consists of a *homogeneous solution* plus a *particular solution*. The *homogeneous solution* is also called the *complementary solution.*

Homogeneous Equation

$$b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \ldots. + b_1 \frac{dy}{dt} + b_0 y = 0$$

Homogeneous Solution

$$y = Ce^{pt}$$

$$\frac{dy}{dt} = pCe^{pt}$$

$$\frac{d^2y}{dt^2} = p^2Ce^{pt}$$

$$\frac{d^my}{dt^m} = p^mCe^{pt}$$

Characteristic Equation

Substitute the form on the previous slide in the DE and cancel the common exponential factor. The result is the characteristic equation shown below.

$$b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p + b_0 = 0$$

Homogeneous Solution Form

The *m* roots of the characteristic equation are determined, and the form of the homogeneous solution for *non-repeated* roots is shown below. Note that if $f(t) = 0$, this result is the complete solution.

$$y_h = C_1 e^{p_1 t} + C_2 e^{p_2 t} + \dots + C_m e^{p_m t}$$

Particular Solution

The particular solution depends on the form of *f*(*t*). Assuming non-repeated roots, the table below shows the forms involved.

Form of $f(t)$	Form assumed for y_p
K	A
Kt	$A_1 t + A_0$
Kt^2	$A_2 t^2 + A_1 t + A_0$
$K_1 \cos \omega t$ and/or $K_2 \sin \omega t$	$A_1 \sin \omega t + A_2 \cos \omega t$
$Ke^{-\alpha t}$	$Ae^{-\alpha t}$

Combining Particular and Homogeneous Solutions

1. The form of the particular solution is substituted in the DE and its constants are determined.
2. The homogeneous and particular solutions are combined and the arbitrary constants from homogeneous solution are determined from boundary or initial conditions.

Example 9-5. Solve DE given below.

$$\frac{dy}{dt} + 2y = 0 \qquad y(0) = 10$$

$$p + 2 = 0 \qquad p = -2$$

$$y = Ce^{-2t} \qquad 10 = Ce^{-0} = C$$

$$y = 10e^{-2t}$$

Example 9-6. Solve DE given below.

$$\frac{dy}{dt} + 2y = 12 \qquad y(0) = 10 \qquad y_h = Ce^{-2t}$$

$$y_p = A \qquad 0 + 2A = 12 \qquad A = 6$$

$$y_p = 6 \qquad y = y_h + y_p = Ce^{-2t} + 6$$

$$10 = Ce^{-0} + 6 = C + 6 \qquad C = 4$$

$$y = 4e^{-2t} + 6$$

Example 9-7. Solve DE given below.

$$\frac{dy}{dt} + 2y = 12\sin 4t \qquad y(0) = 10$$

$$y_h = Ce^{-2t}$$

$$y_p = A_1 \sin 4t + A_2 \cos 4t$$

$$\frac{dy_p}{dt} = 4A_1 \cos 4t - 4A_2 \sin 4t$$

Example 9-7. Continuation.

$$4A_1 \cos 4t - 4A_2 \sin 4t + 2\left(A_1 \sin 4t + A_2 \cos 4t\right) = 12 \sin 4t$$

$$\left(4A_1 + 2A_2\right)\cos 4t + \left(2A_1 - 4A_2\right)\sin 4t = 12 \sin 4t$$

$$4A_1 + 2A_2 = 0$$

$$2A_1 - 4A_2 = 12$$

$$A_1 = 1.2 \qquad A_2 = -2.4$$

Example 9-7. Continuation.

$$y_p = 1.2\sin 4t - 2.4\cos 4t$$

$$y = Ce^{-2t} + 1.2\sin 4t - 2.4\cos 4t$$

$$10 = Ce^{-0} + 1.2\sin(0) - 2.4\cos(0) = C + 0 - 2.4(1)$$

$$C = 12.4$$

$$y = 12.4e^{-2t} + 1.2\sin 4t - 2.4\cos 4t$$

Example 9-8. Solve DE given below.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

$$y(0) = 10 \text{ and } y'(0) = 0$$

$$p^2 + 3p + 2 = 0 \qquad p_1 = -1 \text{ and } p_2 = -2$$

$$y = C_1e^{-t} + C_2e^{-2t}$$

$$\frac{dy}{dt} = -C_1e^{-t} - 2C_2e^{-2t}$$

Example 9-8. Continuation.

$$10 = C_1 + C_2$$

$$0 = -C_1 - 2C_2$$

$$C_1 = 20 \qquad C_2 = -10$$

$$y = 20e^{-t} - 10e^{-2t}$$

Example 9-9. Solve DE given below.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 24 \qquad y(0) = 10 \text{ and } y'(0) = 0$$

$$y_h = C_1e^{-t} + C_2e^{-2t} \qquad y_p = A$$

$$0 + 0 + 2A = 24 \qquad A = 12$$

$$y = C_1e^{-t} + C_2e^{-2t} + 12$$

$$\frac{dy}{dt} = -C_1e^{-t} - 2C_2e^{-2t}$$

Example 9-9. Continuation.

$$10 = C_1 + C_2 + 12$$

$$0 = -C_1 - 2C_2$$

$$C_1 = -4 \qquad C_2 = 2$$

$$y = -4e^{-t} + 2e^{-2t} + 12$$

Some General Properties of Systems Described by CCLODEs

$$\text{homogeneous solution} \Leftrightarrow \text{natural response}$$

$$\text{particular solution} \Leftrightarrow \text{forced response}$$

Stability

A system is said to be stable if its natural response approaches zero as the time increases without limit. If this condition is met, the system will be stable for any finite forcing response. For a stable system, the terms below are often used.

natural response $\Leftrightarrow$ transient response

forced response $\Leftrightarrow$ steady-state response

Classification of Roots of the Characteristic Equation

1. first-order and real
2. first-order and complex (including purely imaginary)
3. multiple-order and real
4. multiple-order and complex (including purely imaginary)

Example 9-10. Investigate properties of DE below.

$$2\frac{d^2y}{dt^2}+10\frac{dy}{dt}+16y=80$$

$$2p^2+10p+16=0 \qquad p_1=-1 \text{ and } p_2=-4$$

$$y_h=C_1e^{-t}+C_2e^{-4t} \qquad y_p=A=5$$

$$y=C_1e^{-t}+C_2e^{-4t}+5$$

Example 9-10. Continuation.

The system is *stable* since both of the terms in the homogeneous solution approach zero as time increases.

Since the system is stable, the *natural response* is a *transient response*, and the *forced response* is a *steady-state response*.

Second-Order Systems

$$b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y = f(t)$$

$$b_2 p^2 + b_1 p + b_0 = 0$$

There are three cases: (1)roots are real and different, (2) roots are real and equal, and (3) roots are complex (including purely imaginary).

Three Forms for Stable Systems

$$y_h = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t}$$

$$y_h = (C_0 + C_1 t) e^{-\alpha t}$$

$$y_h = C_1 e^{-\alpha t} \sin \omega t + C_2 e^{-\alpha t} \cos \omega t$$

Relative Damping

1. If the roots are *real and unequal*, the system is said to be *overdamped.*
2. If the roots are *real and equal*, the system is said to be *critically damped*.
3. If the roots are *complex*, the system is said to be *underdamped.*
4. A special case of an *underdamped* system is when there is no damping. the system is then said to be *undamped.*

Example 9-11. Solve DE given below.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

$$y(0) = 0 \text{ and } y'(0) = 10$$

$$p^2 + 2p + 5 = 0$$

$$p_1, p_2 = -1 \pm 2i$$

$$y = C_1 e^{-t} \sin 2t + C_2 e^{-t} \cos 2t$$

Example 9-11. Continuation.

$$\frac{dy}{dt} = 2C_1 e^{-t} \cos 2t - C_1 e^{-t} \sin 2t$$

$$-2C_2 e^{-t} \sin 2t - C_2 e^{-t} \cos 2t$$

$$0 = 0 + C_2 \qquad 10 = 2C_1 - 0 - 0 - C_2$$

$$C_2 = 0 \text{ and } C_1 = 5$$

$$y = 5e^{-t} \sin 2t$$

Example 9-12. Solve DE given below.`

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 20$$

$$y(0) = 0 \text{ and } y'(0) = 10$$

$$y_h = C_1 e^{-t} \sin 2t + C_2 e^{-t} \cos 2t$$

$$y_p = A = 4$$

$$y = C_1 e^{-t} \sin 2t + C_2 e^{-t} \cos 2t + 4$$

Example 9-12. Continuation.

$$\frac{dy}{dt} = 2C_1 e^{-t}\cos 2t - C_1 e^{-t}\sin 2t$$

$$-2C_2 e^{-t}\sin 2t - C_2 e^{-t}\cos 2t$$

$$0 = 0 + C_2 + 4 \qquad 10 = 2C_1 - 0 - 0 - C_2$$

$$C_2 = -4 \qquad C_1 = 3$$

$$y = 3e^{-t}\sin 2t - 4e^{-t}\cos 2t + 4$$

Chapter 11
Solution of Differential Equations with MATLAB

MATLAB has some powerful features for solving differential equations of all types. We will explore some of these features for the CCLODE forms. The approach here will be that of the Symbolic Math Toolbox. The result will be the form of the function and it may be readily plotted with MATLAB.

Symbolic Differential Equation Terms

y	y
$\frac{dy}{dt}$	Dy
$\frac{d^2y}{dt^2}$	D2y
$\frac{d^ny}{dt^n}$	Dny

Representative CCLODE Form

$$b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y = A \sin at$$

$$y(0) = C_1 \qquad \text{and}\; y'(0) = C_2$$

```
>> y = dsolve('b2*D2y+b1*D1y+b0*y=A*sin(a*t)',
        'y(0)=C1', 'Dy(0)=C2')

>> ezplot(y, [t1 t2])
```

Example 11-1. Solve DE below with MATLAB.

$$\frac{dy}{dt} + 2y = 12 \qquad y(0) = 10$$

```
>> y = dsolve('Dy + 2*y = 12', 'y(0)=10')
y =
6+4*exp(-2*t)

>> ezplot(y, [0 3])
>> axis([0 3 0 10])
```

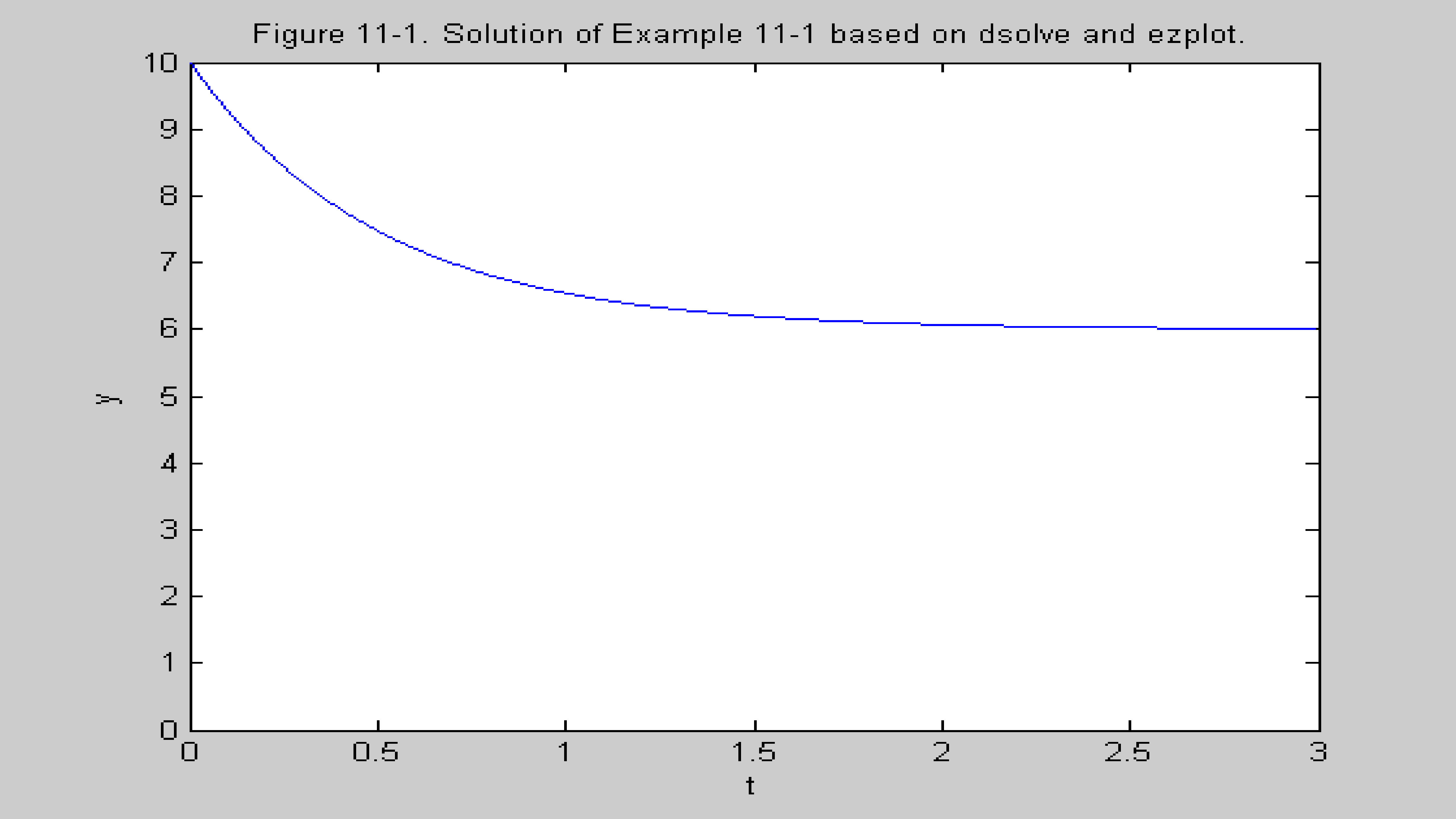


Figure 11-1. Solution of Example 11-1 based on dsolve and ezplot.
10
9
8
7
6
5
4
3
2
1
0
y
0
0.5
1
1.5
2
2.5
3
t

Example 11-2. Solve DE below with MATLAB.

$$\frac{dy}{dt} + 2y = 12\sin 4t \qquad y(0) = 10$$

```
>> y = dsolve('Dy + 2*y = 12*sin(4*t)', 'y(0)=10')
y =
 -12/5*cos(4*t)+6/5*sin(4*t)+62/5*exp(-2*t)

>> ezplot(y, [0 8])
>> axis([0 8 -3 10])
```

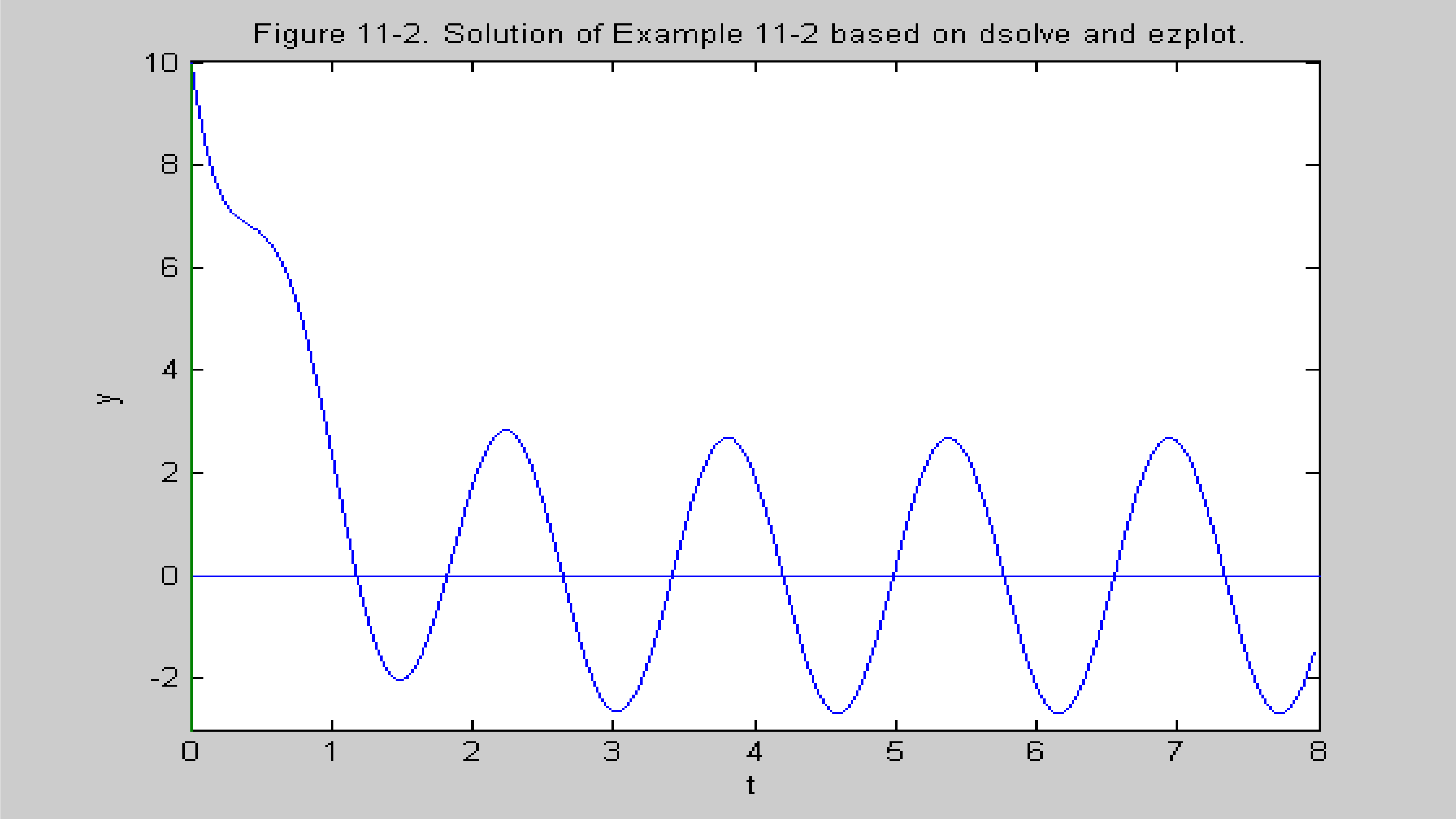


Figure 11-2. Solution of Example 11-2 based on dsolve and ezplot.
y
10
8
6
4
2
0
-2
0
1
2
3
4
5
6
7
8
t

Example 11-3. Solve DE below with MATLAB.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 24$$

$$y(0) = 10 \quad y'(0) = 0$$

```
>> y = dsolve('D2y + 3*Dy + 2*y = 24',
'y(0)=10',  'Dy(0)=0')
y =
12+2*exp(-2*t)-4*exp(-t)

>> ezplot(y, [0 6])
```

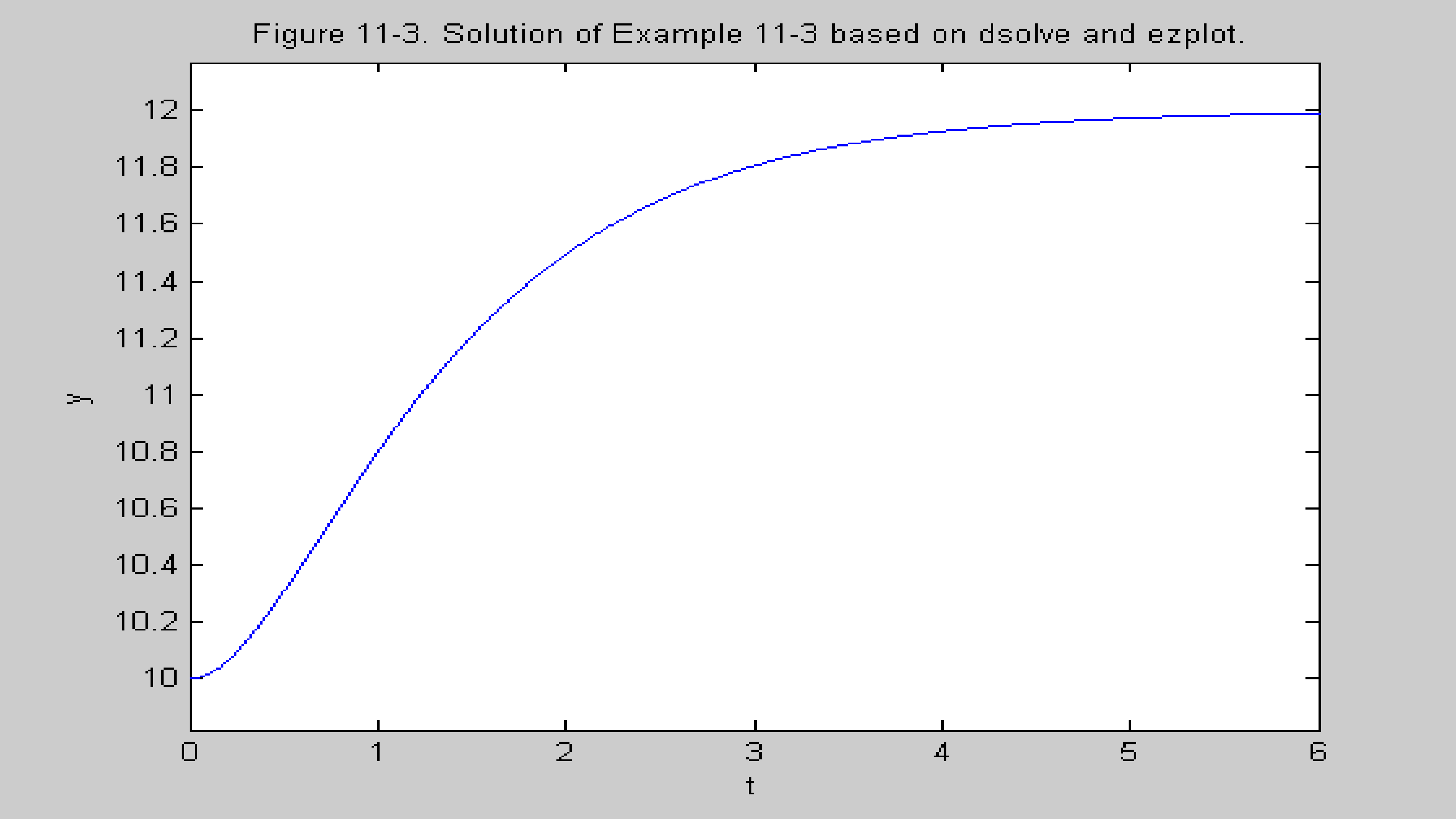


Figure 11-3. Solution of Example 11-3 based on dsolve and ezplot.
12
11.8
11.6
11.4
11.2
11
10.8
10.6
10.4
10.2
10
y
0
1
2
3
4
5
6
t

Example 11-4. Solve DE below with MATLAB.

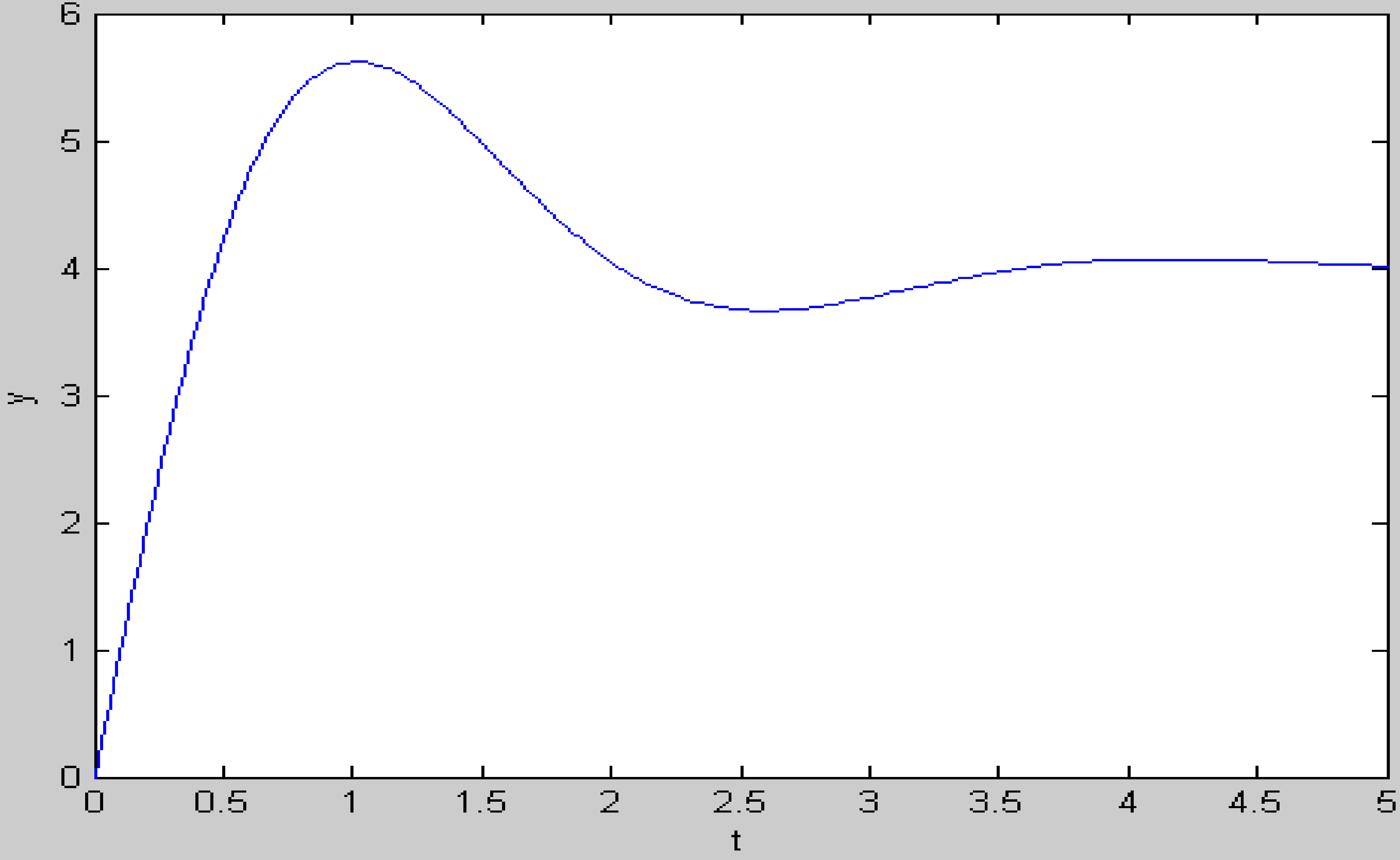
$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 20$$

$$y(0) = 0 \qquad y'(0) = 10$$

```
>> y = dsolve('D2y + 2*Dy + 5*y = 20',
          'y(0) = 0',  'Dy(0) = 10')
y =
4+3*exp(-t)*sin(2*t)-4*exp(-t)*cos(2*t)

>>ezplot(y, [0 5]}
```

Figure 11-4. Solution of Example 11-4 based on dsolve and ezplot.



Symbolic Laplace Transform

Establish t and s as symbolic variables.

```
>> syms t s
```

The time function f is then formed and the Laplace transform command is

```
>> F = laplace(f)
```

Some useful simplifications are

```
>> pretty(F)
>> simplify(F)
```

Symbolic Inverse Laplace Transform

Establish t and s as symbolic variables.

```
>> syms t s
```

The Laplace function F is then formed and the inverse Laplace transform command is

```
>> f = ilaplace(F)
```

The simplification operations may also be useful for inverse transforms.

Example 11-5. Determine the Laplace transform of $f(t)=5t$ with MATLAB.

```
>>syms t s
>> f = 5*t
f =
5*t

>> F = laplace(f)
F =
 5/s^2
```

Example 11-6. Determine the Laplace transform of the function below using MATLAB.

$$v(t) = 3e^{-2t}\sin 5t + 4e^{-2t}\cos 5t$$

```
>> syms t s
>> v = 3*exp(-2*t)*sin(5*t)
        + 4*exp(-2*t)*cos(5*t)
v =
3*exp(-2*t)*sin(5*t)+4*exp(-2*t)*cos(5*t)
```

Example 11-6. Continuation.

```
>> V = laplace(v)
V =
15/((s+2)^2+25)+4*(s+2)/((s+2)^2+25)

>> V=simplify(V)
V =
 (23+4*s)/(s^2+4*s+29)
```

Example 11-7. Determine the inverse transform of the function below using MATLAB.

$$F(s)=\frac{100(s+3)}{(s+1)(s+2)(s^2+2s+5)}$$

```
>> syms t s

>> F=100*(s+3)/((s+1)*(s+2)*(s^2+2*s+5))
F =
(100*s+300)/(s+1)/(s+2)/(s^2+2*s+5)
```

Example 11-7. Continuation.

```
>> f = ilaplace(F)

f =
 50*exp(-t)-20*exp(-2*t)-30*exp(-t)*cos(2*t)-10*exp(-t)*sin(2*t)

>> pretty(f)
     50 exp(-t) - 20 exp(-2 t) - 30 exp(-t) cos(2 t) - 10 exp(-t) sin(2 t)
```

Example 11-8. Determine the inverse transform of the function below using MATLAB.

$$Y(s) = \frac{10}{s+2} + \frac{48}{(s+2)(s^2+16)}$$

```
>> syms t s
>> Y = 10/(s+2) + 48/((s+2)*(s^2+16))
Y =
10/(s+2)+48/(s+2)/(s^2+16)
```

Example 11-8. Continuation.

```
>> y = ilaplace(Y)
y =
62/5*exp(-2*t)-
12/5*cos(16^(1/2)*t)+3/10*16^(1/2)*sin(16^(1/2)*t
)

>> y=simplify(y)
y =
62/5*exp(-2*t)-12/5*cos(4*t)+6/5*sin(4*t)
```

Numerical Differentiation

MATLAB Functions for Numerical Differentiation:

diff()

polyder()

MATLAB is a numerical language and do not perform symbolic mathematics

... well, that is not entirely true because there is “Symbolic Toolbox” available for MATLAB.

```
x = -5:1:5;

% Define the function y(x)
y = x.^3 + 2*x.^2 - x + 3;

% Plot the function y(x)
plot(x,y)
title('y')

% Find nummerical solution to dy/dx
dydx_num = diff(y)./diff(x);

dydx_exact = 3*x.^2 + 4.*x -1;

dydx = [[dydx_num, NaN]', dydx_exact']

% Plot nummerical vs analytical solution to dy/dx
figure(2)
plot(x,[dydx_num, NaN], x, dydx_exact)
title('dy/dx')
legend('numerical solution', 'analytical solution')
```

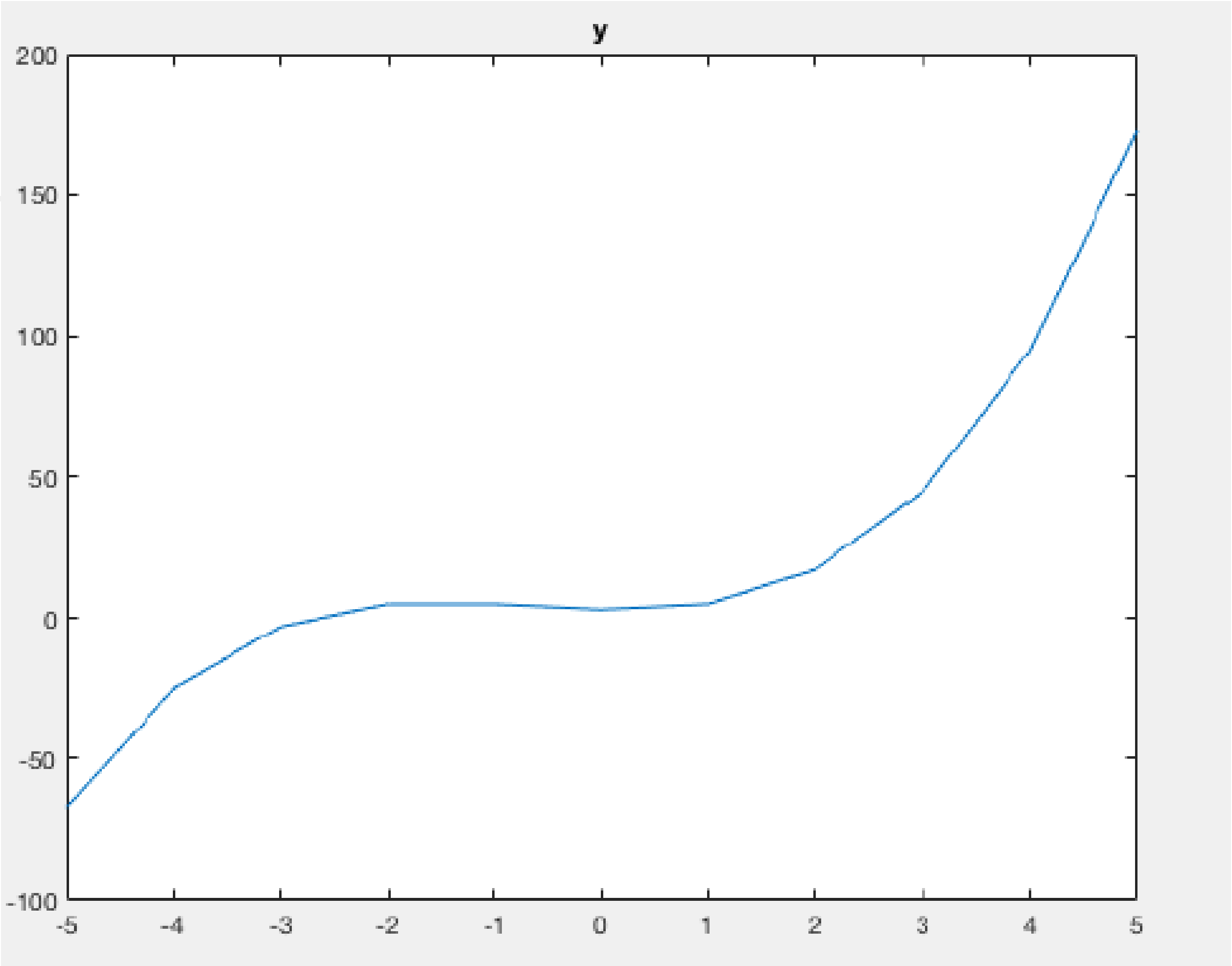
Numerical Solution → Exact Solution →

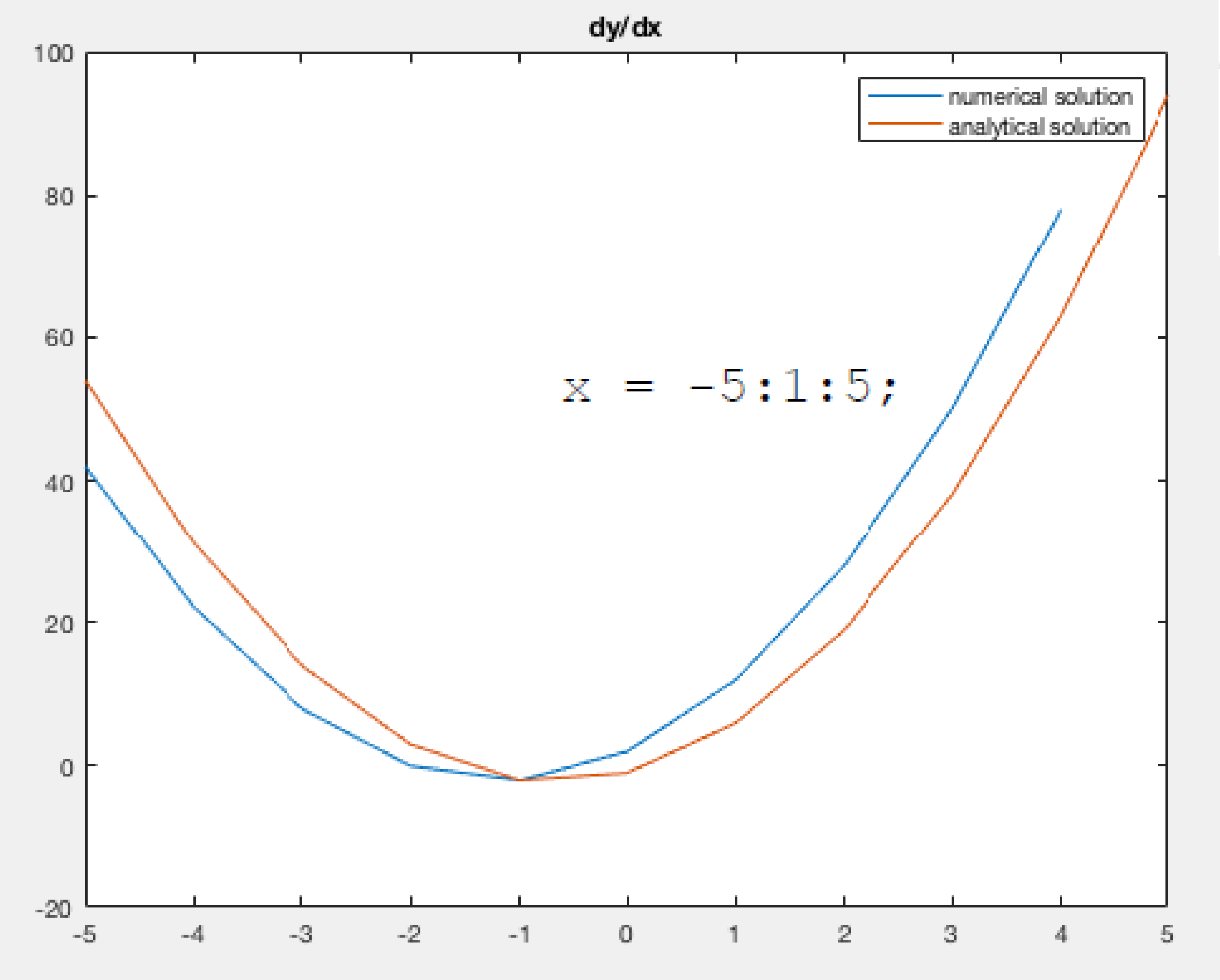
```
dydx =

    42    54
    22    31
     8    14
     0     3
    -2    -2
     2    -1
    12     6
    28    19
    50    38
    78    63
   NaN    94
```

$$y = x^3 + 2x^2 - x + 3$$

$$\frac{dy}{dx} = 3x^2 + 4x - 1$$





```
p = [1 2 -1 3];

polyder(p)
```

$$y = x^3 + 2x^2 - x + 3$$

```
ans =
      3        4       -1
```

$$\frac{dy}{dx} = 3x^2 + 4x - 1$$

Differentiation on Polynomials

Find the derivative for the product:

$$(3x^2 + 6x + 9)(x^2 + 2x)$$

We will use the *polyder(a,b)* function.

Another approach is to use define is to first use the *conv(a,b)* function to find the total polynomial, and then use *polyder(p)* function.

We have that

$$p_1 = 3x^2 + 6x + 9$$

and

$$p_2 = x^2 + 2x$$

The total polynomial becomes then:

$$p = p_1 \cdot p_2 = 3x^4 + 12x^3 + 21x^2 + 18x$$

As expected, the results are the same for the 2 methods used above:

$$\frac{dp}{dx} = \frac{d(3x^4 + 12x^3 + 21x^2 + 18x)}{dx} = 12x^3 + 36x^2 + 42x + 18$$

```
% Define the polynomials
p1 = [3 6 9];
p2 = [1 2 0]; %Note!

% Method 1
polyder(p1,p2)


% Method 2
p = conv(p1,p2)
polyder(p)
```

```
ans =
        12     36     42     18

p =
         3     12     21     18     0

ans =
        12     36     42     18
```